STRUCTURE OF HEAT CONDUCTION

A. G. Shashkov and T. N. Abramenko

UDC 536.24.01

A new structural approach is proposed for analyzing and synthesizing heat-transfer phenomena. The method is illustrated with examples of one-dimensional heat conduction.

The structure of heat transfer should reflect the interrelationships and mutual effects of the spatial–temporal formation of the temperature field and the heat fluxes at the boundary of the object and within it. To a certain extent, but not always in an explicit form, these functions are performed by the differential equations of thermal conductivity with the boundary conditions and their solutions, which can be used to calculate steady-state and transient, local and average temperatures and heat fluxes. However, these results cannot be considered the ultimate goal of a study of heat conduction. They do not permit a detailed analysis of the interrelationship between heat transfer at the boundary of the object and within it, and they are of little use in the solution of engineering problems of such current interest as the correction and synthesis of heat systems with prespecified processes for shaping the temperature fields.

In particular, this ineffectiveness of a direct use of the solutions of the differential equations of heat conduction for analysis and synthesis can be attributed to the circumstance that they are expressed in terms of complicated combinations and derivatives, integrals, and terms containing nonelementary functions. The complexity of these equations masks the physical meaning of the individual terms and their relationships.

To some extent these shortcomings are avoided by using the structural method related to an analysis of the structural diagram constructed on the basis of the Laplace-transformed differential or integral heat-conduction equation with the appropriate boundary conditions.*

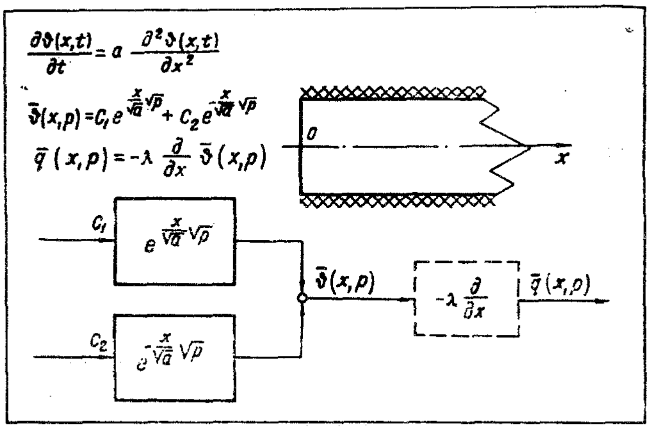

Fig. 1. Structure of the general solution of the one-dimensional heat-conduction problem.

*As a result, the equations convert into simpler equations, and frequently the results are purely algebraic.

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 28, No. 5, pp. 884-893, May, 1975. Original article submitted October 26, 1974.

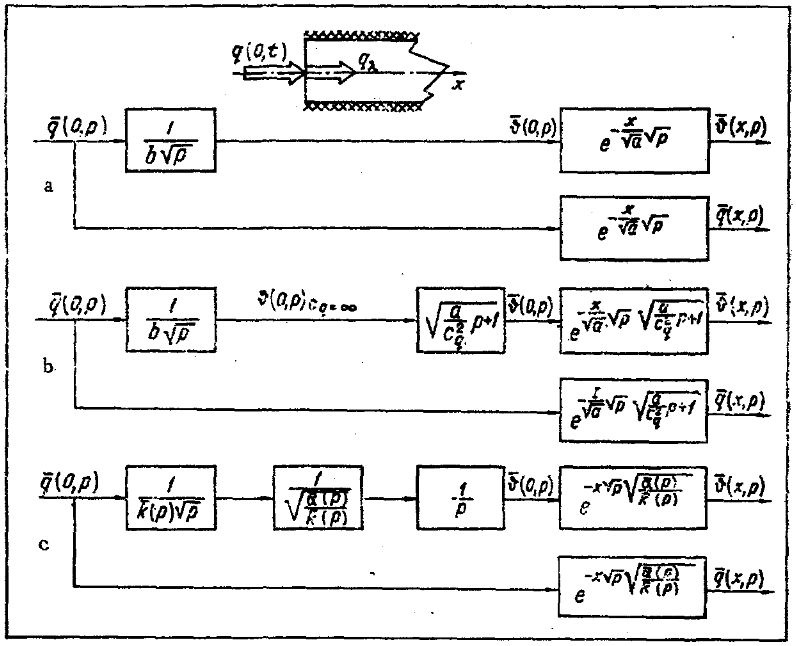

Fig. 2. Structural diagram for heat transfer in a rod. a) Transfer described by the heat-conduction equation; b) by a hyperbolic equation; c) by an integrodifferential equation.

The structural diagram is composed in such a manner that it yields, in accordance with established rules, the product of the transforms of functions describing the individual units, i.e., the transfer functions of the individual units, groups of such units, and the entire system [1].

The Borel theorem can be used to find the inverse transform of a product of transforms from the inverse transforms of the cofactors; i.e., this theorem makes it possible to convert to the timelike region.

The operator functions describing the units must be as simple and physically meaningful as possible. Accordingly, in the construction of the structural diagram, those units whose "output" consists of the temperatures and heat fluxes at the boundary of the media should be distinguished.

The structural diagram composed in this manner reflects the mutual effects (coupling) of the formation of the boundary conditions and the temperature fields and the heat fluxes in the object. On the whole, this diagram compactly codes and correctly describes the functional relationship between the "input" and the "output."

On the basis of this structural diagram it is possible to immediately write out the equation for a unit or for a group of units. In practice this is a very important capability, since it permits us to establish the relationship between the coefficients and the variables of the equations.

The structural approach is apparently the most effective general method for studying (analyzing and synthesizing) complicated interrelated systems containing elements of different physical natures, in particular, electrical and thermal elements. In this sense the structural approach is a methodological bridge connecting the studies of electrical and thermal processes in electrical—thermal systems.*

One particular application of structural analysis, but of practical importance, could be in the development of transient methods for thermal measurements. The structural diagram, as a linear model of the physical process (system), can offer a convenient description and analysis of the interrelated transfer processes (e.g., thermal conductivity and thermodiffusion).

* For example, in a thermal anemometer the sensitive element — a heated filament at which heat is evolved, transferred within the filament, and dissipated in the surrounding medium — is the element of a complicated electronic circuit. For this reason the basic characteristics of the anemometer (its sensitivity and static and dynamic errors) are governed not so much by the properties of the sensitive element as by the structure of the measurement circuit as a whole.

It is not the purpose of a study of the structure of a process or phenomenon to develop a new physical theory or principle but to compare theories and principles, i.e., to determine their relationships, and thereby generalize results.

The formal concepts of the "input" and "output" of the structural diagram are associated with the fundamental concepts of "cause" and "effect," and to avoid confusion we take up these questions.

A transient heat transfer results from a disruption of the steady thermal state of a system. In this sense the cause of the transient heat transfer could be any changes in the boundary and initial conditions.

The subject of a study of transient heat conduction, described by the Fourier law, consists of the synchronized changes and interrelationships between the temperatures and heat fluxes due to a disruption of the equilibrium thermal state of the object. This disruption is treated as the cause of the transient nature of the situation. The temperature gradient within the object can be treated as both a cause and an effect of the heat transfer (heat fluxes within the object). Within the framework of the structural diagram this circumstance is reflected by the circumstance that the structural diagram is invariant with respect to the "input" and the "output." This invariance is a consequence of the Fourier hypothesis (law) $\vec{q} = -\lambda \text{grad} T$, which agrees well with observations when the spatial — temporal scales of the heat-conduction phenomenon are quite large.

For the case in which the rate at which thermal energy is supplied is very large, or for a transient process considered over a short time interval, Morse and Feshbach [2] have proposed the following hyperbolic equation for calculating the heat conduction:

$$\frac{1}{c_q^2} \frac{\partial^2 T}{\partial t^2} + \frac{1}{a} \frac{\partial T}{\partial t} = \nabla T.$$

This equation can be derived by writing the law of heat transfer by conduction in the form

$$-\lambda \text{ grad } T = \vec{q} + \tau_r \frac{\partial \vec{q}}{\partial t},$$

where

$$\tau_r = a/c_q^2.$$

Below we examine the structure of the heat transfer described by a hyperbolic equation.

Materials displaying thermal "memory" or "inheritance" are a topic of much current interest [3, 4]. In such materials the "history" of a process which is occurring significantly influences heat conduction at the time under consideration.

To study the structure of heat transfer for the example of the one-dimensional heat-conduction problem we use the equation

$$\alpha(0) \frac{\partial T(x, t)}{\partial t} + \int_0^\infty \alpha'(s) \frac{\partial T(x, t-s)}{\partial t} ds = k(0) \frac{\partial^2 T(x, t)}{\partial x^2} + \int_0^\infty k'(s) \frac{\partial^2 T(x, t-s)}{\partial x^2} ds,$$

where $\alpha'(s)$ and $k'(s)$, the kernels of the integral operators, are related to the relaxation functions $\alpha(s)$ and $k'(s)$ by

$$\alpha'(s) = \frac{d}{ds} \alpha(s), \quad k'(s) = \frac{d}{ds} k(s);$$

in the limit $s \to \infty$

$$\alpha'(s) \to 0, \quad k'(s) \to 0.$$

The relaxation functions can be approximated by various expressions, whose final form, with the numerical constants, can be determined only experimentally.

STRUCTURE OF ONE-DIMENSIONAL HEAT CONDUCTION

We will first determine the structure of heat transfer for relatively simple examples and then go on to more complicated examples. We consider a very simple example: the heat conduction of a thermally insulated, semi-infinite rod into one end of which there is a heat flux. We show how to determine the structure of heat transfer for the ordinary heat-conduction law $q = -\lambda \partial T/\partial x$, for the modified law $q = -\lambda(\partial t/dx) - (a/cq^2) - (\partial q/\partial t)$, and for the case in which memory effects are taken into account. We then make a comparative structural analysis.

I. Let us determine the structure of heat transfer in a semi-infinite rod whose lateral surface is thermally insulated and into one end of which there is a heat flux which varies with the time. We do not take up the cause of the change in the heat flux at this point. At this point the only important consideration is that the heat flux varies with the time. We denote the rod temperature by $T(x, t)$, and we denote the temperature of the surrounding medium by θ. The problem is solved by Laplace transforms. The problem is formulated analytically as follows:

$$\frac{\partial \vartheta(x, t)}{\partial t} = a\frac{\partial^2 \vartheta(x, t)}{\partial x^2}, \quad \vartheta(x, 0) = 0, \quad \vartheta(\infty, t) = 0,$$

$$-\lambda \left.\frac{\partial \vartheta(x, t)}{\partial x}\right|_{x=0} = q(t);$$

where $\vartheta(x, t) = T(x, t) - \theta$, $\theta = \text{const}$.

We denote the Laplace transform of $\vartheta(x, t)$ by

$$L\{\vartheta(x, t)\} = \int_0^\infty \vartheta(x, t)e^{-pt}dt = \bar{\vartheta}(x, p).$$

We write the general solution of the equation as

$$\bar{\vartheta}(x, p) = C_1 e^{\frac{x}{\sqrt{a}}\sqrt{p}} + C_1 e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

The general solution of the equation for the Laplace transform of the temperature $\bar{\vartheta}(x, p)$ is described by the structural diagram in Fig. 1. The solution is simple in form, so that the structural diagram representing this solution carries almost no new information about the structure of heat transfer in the rod. However, this diagram does emphasize that the arbitrary constants C_1 and C_2 are inputs or agents and must be specified, i.e., determined from the boundary conditions.

It can be shown that we have $C_1 = 0$ and $C_2 = \bar{q}(0, p)/b\sqrt{p}$, where $b = \lambda/\sqrt{a}$.

A solution of the equation satisfying the boundary conditions is

$$\bar{\vartheta}(x, p) = \bar{q}(p)\frac{1}{b\sqrt{p}}e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

Since

$$\bar{\vartheta}(0, p) = \bar{q}(p)\frac{1}{b\sqrt{p}},$$

then

$$\bar{\vartheta}(x, p) = \bar{\vartheta}(0, p)e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

The flux at cross section x, $\bar{q}(x, p)$, is given by

$$\bar{q}(x, p) = -\lambda\frac{\partial \bar{\vartheta}(x, p)}{\partial x} = \bar{q}(p)e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

On the basis of the equations for $\bar{\vartheta}(x, p)$ and $\bar{q}(x, p)$ we can construct a structural diagram (Fig. 2a) which displays the structure of the heat transfer.

The structural diagram is shown as two parallel circuits, one consisting of two units and the other of a single unit. The diagram has a single input and two outputs, $\bar{\vartheta}(x, p)$ and $\bar{q}(x, p)$.

We note that the order in which the units are connected could be changed, and the directions in which the units affect each other could also be changed. If these changes are made, of course, the transfer functions of the units should be replaced by their inverses. This is possible because of the synchronous nature of the formation of the temperature fields and heat fluxes, which is a consequence of the Fourier law which we have adopted.

Examining the structural diagram in Fig. 2, we find that it can be used to solve several direct and inverse problems of practical interest (e.g., it can be used to develop methods for determining thermal properties of materials).

A. If some time dependence q(t) of the heat flux is specified, the temperature field of the rod can be found.

B. The time dependence q(t) providing a desired change of the rod temperature, in particular, of the temperature at the end of the rod, can be found.

C. The thermal properties of the rod can be found from the temperature field, itself determined experimentally.

To find $\bar{q}(p)$ we should specify the method by which the heat is supplied to the end of the rod (conduction, convection, or radiation) and then formulate the analytic dependence of the heat flux on the temperature on the end of the rod, taking into account the heat-transfer mechanism at the surface.

II. Let us determine the structure of heat transfer in a semi-infinite rod, whose lateral surface is thermally insulated and at whose end a time-varying heat flux is applied. In contrast with the preceding case we assume that the heat conduction in the rod is described by a hyperbolic equation.

The mathematical formulation of this problem is

$$\frac{1}{c_q^2}\frac{\partial^2\vartheta}{\partial t^2}+\frac{1}{a}\frac{\partial\vartheta}{\partial t}=\frac{\partial^2\vartheta}{\partial x^2},$$

$$\vartheta(x,\ 0)=0,\quad \frac{\partial\vartheta(x,\ 0)}{\partial t}=0,\quad \vartheta(\infty,\ t)=0,$$

$$-\lambda\left.\frac{\partial\vartheta(x,\ t)}{\partial x}\right|_{x=0}-\frac{a}{c_q^2}\left.\frac{\partial q(x,\ t)}{\partial t}\right|_{x=0}=q(0,\ t),\quad q(0,\ 0)=0.$$

The solution for the transform of the temperature is

$$\bar{\vartheta}(x,\ p)=\bar{q}(0,\ p)\frac{1}{b\sqrt{p}}\sqrt{\frac{a}{c_q^2}p+1}\, e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

Setting x = 0, we find

$$\bar{\vartheta}(0,\ p)=\bar{q}(0,\ p)\frac{1}{b\sqrt{p}}\sqrt{\frac{a}{c_q^2}p+1}.$$

We then find

$$\bar{\vartheta}(x,\ p)=\bar{\vartheta}(0,\ p)\,e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

We solve the equation

$$\bar{q}(x,\ p)=-\lambda\frac{\partial\bar{\vartheta}(x,\ p)}{\partial x}-\frac{a}{c_q^2}p\bar{q}(x,\ p)$$

for

$$\bar{q}(x,\ p)=-\lambda\frac{\partial\bar{\vartheta}(x,\ p)}{\partial x}\frac{1}{\frac{a}{c_q^2}p+1}=\bar{q}(0,\ p)\,e^{-\frac{x}{\sqrt{a}}\sqrt{p}}.$$

Using the expressions for $\bar{\vartheta}(x,\ p)$ and $\bar{q}(x,\ p)$ we can construct the structural diagram (Fig. 2b). Comparison of the structural diagrams in Fig. 2a and Fig. 2b shows that in the case of a finite heat-propagation velocity a new unit appears in the diagram, described by the operator $\sqrt{a/c_q^2(p+1,}$)which takes into account the effect of the finite heat-propagation velocity on the shaping of the temperature field in the rod. The operator corresponding to the propagation of the temperature (heat flux) in the rod contains the additional factor $\sqrt{a/c_q^2(p+1)}$ in the argument of an exponential function; the operators corresponding to the units of the structural diagram contain the constant quantities a, b, and c_q, which describe the heat conduction in the medium. Accordingly, the structural diagram can be used to point out ways to determine these constants, including the heat-propagation velocity. Let us consider a few of these ways.

The coefficients a and b are known. The behavior of the heat flux, q(0, t), is determined by the experimental program. Accordingly, the transform of the heat flux $\bar{q}(0, p)$, is specified.

We use the operators corresponding to the units (Fig. 2b) which establish the relationship between $\bar{q}(0, p)$ and $\bar{\vartheta}(0, p)$; then we have

$$\bar{\vartheta}(0, p)|_{c_q=\infty} = \bar{q}(0, p)\frac{1}{b\sqrt{p}}, \quad \vartheta(0, t)|_{c_q=\infty} = L^{-1}\left\{\bar{q}(0, p)\frac{1}{b\sqrt{p}}\right\},$$

$$\bar{\vartheta}(0, p) = \bar{\vartheta}(0, p)|_{c_q=\infty}\sqrt{\frac{a}{c_q^2}p + 1},$$

$$\vartheta(0, t) = L^{-1}\left\{\bar{\vartheta}(0, p)|_{c_q=\infty}\sqrt{\frac{a}{c_q^2}p + 1}\right\}.$$

Since

$$L^{-1}\left\{\frac{1}{b\sqrt{p}}\right\} = \frac{1}{b\sqrt{\pi}}\frac{1}{\sqrt{t}},$$

$$L^{-1}\left\{\sqrt{\frac{a}{c_q^2}p + 1}\right\} = -\frac{\sqrt{a}}{c_q}\cdot\frac{1}{2\sqrt{\pi}}\cdot\frac{e^{-\frac{c_q^2}{a}\tau}}{\tau\sqrt{\tau}},$$

we can write the following equations, making use of the convolution theorem:

$$\vartheta(0, t)|_{c_q=\infty} = \int_0^t q(0, t-\tau)\frac{1}{b\sqrt{\pi\tau}}d\tau,$$

$$\vartheta(0, t) = -\int_0^t \vartheta(0, t-\tau)|_{c_q=\infty}\frac{1}{2\sqrt{\pi}}\frac{\sqrt{a}}{c_q}\frac{e^{-\frac{c_q^2}{a}t}}{t\sqrt{t}}d\tau.$$

Calculating the quantity $\vartheta(0, t)|_{c_q=\infty}$ for time t, measuring the temperature $\vartheta(0, t)$ at this time, and solving (analytically or graphically) the system of equations for a/c_q^2, we can determine its value.

This is the fundamental basis for determining the heat-propagation velocity. On this basis it is possible to develop several procedures, using a variety of laws governing the change in the heat flux. For example, frequency methods, based on a periodic change of q(0, t) (in particular, a sinusoidal variation), can be developed. To obtain a variety of types of information we can use the operator corresponding to the unit relating $\bar{\vartheta}(0, p)$ and $\bar{\vartheta}(x, p)$:

$$\bar{\vartheta}(x, p) = \bar{\vartheta}(0, p)\, e^{-\frac{x}{\sqrt{a}}\sqrt{p}}\sqrt{\frac{a}{c_q^2}p + 1}.$$

Since [5]

$$L^{-1}\left\{e^{-\frac{x}{\sqrt{a}}\sqrt{p}}\sqrt{\frac{a}{c_q^2}p + 1}\right\} = e^{-\frac{c_q}{2a}x}\delta\left(t - \frac{x}{c_q}\right) + x\frac{c_q}{2a}e^{-\frac{c_q}{2a}t}\frac{I_1\left(\frac{c_q^2}{2a}\sqrt{t^2 - \left(\frac{x}{c_q}\right)^2}\right)}{\sqrt{t^2 - \left(\frac{x}{c_q}\right)^2}},$$

we find, using the convolution theorem,

$$\vartheta(x, t) = \int_0^t \vartheta(0, t-\tau)\left[e^{-x\frac{c_q}{2a}}\delta\left(\tau - \frac{x}{c_q}\right) + x\frac{c_q}{2a}e^{-\frac{c_q^2}{2a}\tau}\frac{I_1\left(\frac{c_q^2}{2a}\sqrt{\tau^2 - \left(\frac{x}{c_q}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{x}{c_q}\right)^2}}\right]d\tau.$$

Evaluating the integral and substituting in the measured values of $\vartheta(x, t)$ and $\vartheta(0, t)$, we can determine a/c_q^2.

III. We now determine the structure of heat transfer in a semi-infinite rod thermally insulated on its lateral surface, with a time-varying heat flux at one of its ends.

The heat conduction in the rod is described by an integrodifferential equation incorporating memory. The analytic formulation of the problem is

$$\alpha(0)\frac{\partial\vartheta(x,t)}{\partial t}+\int_0^\infty\alpha'(s)\frac{\partial\vartheta(x,t-s)}{\partial t}ds=k(0)\frac{\partial^2\vartheta(x,t)}{\partial x^2}+\int_0^\infty k'(s)\frac{\partial^2\vartheta(x,t-s)}{\partial x^2}ds,$$

$$\vartheta(x,0)=0,\quad \frac{\partial\vartheta(x,0)}{\partial t}=0,\quad \vartheta(\infty,t)=0,$$

$$-k(0)\frac{\partial\vartheta(x,t)}{\partial x}\Big|_{x=0}-\int_0^\infty k'(s)\frac{\partial\vartheta(x,t-s)}{\partial x}ds\Big|_{x=0}=q(0,t),$$

$$\vartheta=T(x,t)-\theta,\quad \theta=\mathrm{const}.$$

Using the Borel theorem and taking into account the homogeneous initial conditions, we find

$$\alpha(0)p\bar\vartheta(x,p)+[p\bar\alpha(p)-\alpha(0)]p\bar\vartheta(x,p)$$
$$=k(0)\frac{\partial^2\bar\vartheta(x,p)}{\partial x^2}+[p\bar k(p)-k(0)]\frac{\partial^2\bar\vartheta(x,p)}{\partial x^2},\quad \bar\vartheta(\infty,p)=0,$$

$$-k(0)\frac{\partial\bar\vartheta(x,p)}{\partial x}\Big|_{x=0}-[p\bar k(p)-k(0)]\frac{\partial\bar\vartheta(x,p)}{\partial x}\Big|_{x=0}=\bar q(0,p).$$

The equation for the transform of the temperature is

$$\frac{\partial^2\bar\vartheta(x,p)}{\partial x^2}-p\frac{\bar\alpha(p)}{\bar k(p)}\bar\vartheta(x,p)=0.$$

Its solution satisfying the boundary conditions is

$$\bar\vartheta(x,p)=\bar q(0,p)\frac{1}{p\sqrt{p}\,\bar k(p)\sqrt{\bar\alpha(p)/\bar k(p)}}e^{-x\sqrt{p}\sqrt{\frac{\bar\alpha(p)}{\bar k(p)}}}.$$

Since

$$\bar\vartheta(0,p)=\bar q(0,p)\frac{1}{p\sqrt{p}\,\bar k(p)\sqrt{\bar\alpha(p)/\bar k(p)}},$$

then

$$\bar\vartheta(x,p)=\bar\vartheta(0,p)e^{-x\sqrt{p}\sqrt{\bar\alpha(p)/\bar k(p)}}.$$

We find an equation for $\bar q(x,p)$:

$$\bar q(x,p)=-p\bar k(p)\frac{\partial\bar\vartheta(x,p)}{\partial x}=\bar q(0,p)e^{-x\sqrt{p}\sqrt{\bar\alpha(p)/\bar k(p)}}.$$

Using the expressions for $\bar\vartheta(x,p)$ and $\bar q(x,p)$ we can construct the structural diagram of Fig. 2c.

Comparative analysis of the structure of the heat transfer for the ordinary Fourier equation, for a hyperbolic equation, and for an equation incorporating memory shows that the structure of the heat transfer is strongly influenced by the nature of the kernels $\alpha'(s)$ and $k'(s)$ of the integral operators. In particular, for $k'(s)=0$ we find $k(s)=\mathrm{const}=k(0)=\lambda U(t)$, where $U(t)$ is the unit step function,

$$U(t)=\begin{cases}0, & \text{for } t<0,\\ 1, & \text{for } t\geqslant 0.\end{cases}$$

We thus have

$$\bar k(p)=\lambda\frac{1}{p}.$$

If we assume $\alpha'(s)=0$ and thus $\alpha(s)=\mathrm{const}=\alpha(0)=\rho c U(t)$, we find $\bar\alpha(p)=\rho c(1/p)$.

Substituting $\bar\alpha(p)=\rho c/p$ and $\bar k(p)=\lambda/p$ into the corresponding operator functions of the units of the structural diagram in Fig. 2c, we find that the structure of heat transfer for such kernels is analogous to that in the case in which there is no relaxation of the properties of the medium.

Comparing the structural diagrams in Fig. 2b and Fig. 2c, we see that, when the transforms of the relaxation functions are

$$\bar{\alpha}(p)=\frac{\rho c}{p}\,,\quad \bar{k}(p)=\frac{\lambda}{p\left(\frac{a}{c^2}\,p+1\right)}\,,$$

$$L^{-1}\{\bar{\alpha}(p)\}=L^{-1}\left\{\frac{\rho c}{p}\right\}=\rho c U(t)=\alpha(t),$$

$$L^{-1}\{\bar{k}(p)\}=L^{-1}\left\{\frac{\lambda}{p\left(\frac{a}{c^2}\,p+1\right)}\right\}=\lambda\left(1-a^{-\frac{c^2}{a}t}\right)=k(t),$$

the structural diagram of heat transfer with memory effects acquires the form of the structural diagram for heat transfer described by a hyperbolic heat-conduction equation.

Comparison of the structures in Fig. 2b and Fig. 2c thus leads to the conclusion that the hyperbolic heat-conduction equation describes heat transfer in media which have an ideal thermal elasticity, i.e., for which the temperature is taken on instantaneously ($\alpha(t)=\rho c U(t)$) and in which the heat flux is dissipated over time as the result of an exponential relaxation of the thermal conductivity, with a time constant a/c^2.

LITERATURE CITED

1. A. A. Voronov, Elements of the Theory of Automatic Control [in Russian], Voenizdat, Moscow (1954).
2. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Vols. 1 and 2, McGraw-Hill, New York (1953).
3. I. M. Nunzioto, "On heat conduction in materials with memory," Quart. Appl. Math., 187 (July, 1971).
4. A. V. Lykov, Inzh.-Fiz. Zh., 26, No. 5 (1974).
5. G. Doetsch, Guide to the Application of Laplace and Z Forms, Van Nostrand Reinhold.